Homogenization of semi-linear PDEs with discontinuous effective coefficients

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Abstract

We study the asymptotic behavior of solution of semi-linear PDEs. Neither periodicity nor ergodicity will be assumed. In return, we assume that the coefficients admit a limit in Cèsaro sense. In such a case, the averaged coefficients could be discontinuous. We use probabilistic approach based on weak convergence for the associated backward stochastic differential equation in the S-topology to derive the averaged PDE. However, since the averaged coefficients are discontinuous, the classical viscosity solution is not defined for the averaged PDE. We then use the notion of "L^p-viscosity solution" introduced in [6]. We use BSDEs techniques to establish the existence of L^p-viscosity solution for the averaged PDE. We establish weak continuity for the flow of the limit diffusion process and related the PDE limit to the backward stochastic differential equation via the representation of L^p-viscosity solution.

Keys words: *Backward stochastic differential equations (BSDEs), L^p-viscosity solution for PDEs, homogenization, S-topology, limit in Cèsaro sense.*
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1 Introduction

Homogenization of a partial differential equation (PDE) is the process of replacing rapidly varying coefficients by new ones such that the solutions are close. Example: Let a be a

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one dimensional periodic function which is uniformly elliptic. For $\varepsilon > 0$, we consider the operator

$$L_\varepsilon = div(a(\frac{x}{\varepsilon})\nabla)$$

For small ε, L_ε can be replaced by

$$L = div(\overline{a}\nabla)$$

where $\overline{a}$ is the averaged (or limit or effective) coefficient associated to a. As $\varepsilon \to 0$, solutions of parabolic equations

$$\partial_t u = L_\varepsilon u, \qquad u(0, x) = f(x)$$

are close to the corresponding solutions with L_ε replaced by L.

The probabilistic approach to homogenization gives a good description of this topic in the periodic or ergodic case. It is based on the asymptotic analysis of the diffusion process associated to the operator L_ε. The averaged coefficient $\overline{a}$ is then determined as a certain "mean" of a with respect to the invariant probability measure of the diffusion process associated to L.

There is a vast literature on the homogenization of PDEs with periodic coefficients, see for example monographs [2, 10, 19] and the references therein. There also exists a considerable literature on the study of asymptotic analysis of stochastic differential equations (SDEs) with periodic structures and its connection with homogenization of second order partial differential equations (PDEs). Actually, coming from relations with semilinear and/or quasilinear PDEs given by a generalized Feynman-Kac formula, forward-backward SDEs (FBSDEs) have also been considered, see among others [3, 4, 5, 8, 9, 11, 17, 20, 21] and the references therein.

In the case where the periodicity and the ergodicity are not assumed, we don't have enough information about the invariant probability measure and hence, the situation is more delicate.

In [13], Khasminskii & Krylov have considered the averaging of the following family of diffusions process

$$\begin{cases} x_t^{1,\varepsilon} = x_1 + \frac{1}{\varepsilon}\int_0^t \varphi(x_s^{1,\varepsilon}, x_s^{2,\varepsilon})dW_s \\ \\ x_t^{2,\varepsilon} = x_2 + \int_0^t b^{(1)}(x_s^{1,\varepsilon}, x_s^{2,\varepsilon})ds + \int_0^t \sigma^{(1)}(x_s^{1,\varepsilon}, x_s^{2,\varepsilon})d\widetilde{W}_s \end{cases} \tag{1.1}$$

where $x_t^{1,\varepsilon}$ is a null-recurrent fast component and $x_t^{2,\varepsilon}$ is a slow component. The function φ (resp. σ) is $\mathbb{R}$-valued (resp. $\mathbb{R}^{d\times(k-1)}$-valued). $(W, \widetilde{W})$ is a $\mathbb{R}^k$-dimensional standard Brownian motion which components W is one dimensional while $\widetilde{W}$ is $\mathbb{R}^{k-1}$-dimensional. They then studied the averaging of system (1.1). They defined the averaged coefficients as a limit in Ȅesaro sense. With the additional assumption that the presumed SDE limit is weakly unique, they proved that the process $(\varepsilon x_t^{1,\varepsilon}, x_t^{2,\varepsilon})$ converges in distribution towards a Markov diffusion (X_t^1, X_t^2). As a byproduct, they derived the limit behavior for the linear PDE associated to $(\varepsilon x_t^{1,\varepsilon}, x_t^{2,\varepsilon})$, in the case where the weak uniqueness holds in the Sobolev space $W_{d+1,loc}^{1,2}$.

In the present note, exploiting the idea of [13], we study the homogenization of a parabolic semilinear PDE in the case where both the periodicity and the ergodiciy are not be assumed. We define the averaged coefficients as a limits in Ȅesaro sense. In such a way, the limit coefficients could be discontinuous. More precisely, we consider the following sequence of semi-linear PDEs, indexed by $\varepsilon > 0$,

$$\begin{cases} \frac{\partial v^\varepsilon}{\partial s}(s,\, x_1,\, x_2) = \mathcal{L}^\varepsilon(x_1,\, x_2) v^\varepsilon(s, x_1,\, x_2) + f(\frac{x_1}{\varepsilon},\, x_2,\, v^\varepsilon(s,\, x_1,\, x_2)), \quad s \in (0,\, t) \\ \\ v^\varepsilon(0,\, x_1,\, x_2) = H(x_1,\, x_2) \end{cases} \tag{1.2}$$

where

$$\mathcal{L}^\varepsilon(x_1,\, x_2) \;=\; a_{00}(\frac{x_1}{\varepsilon},\, x_2)\frac{\partial^2}{\partial^2 x_1} + \sum_{i,\,j=1}^{d} a_{ij}(\frac{x_1}{\varepsilon},\, x_2)\frac{\partial^2}{\partial x_{2i}\partial x_{2j}} + \sum_{i=1}^{d} b_i^{(1)}(\frac{x_1}{\varepsilon},\, x_2)\frac{\partial}{\partial x_{2i}},$$

and the real valued measurable functions f and H are defined on $\mathbb{R}^{d+1} \times \mathbb{R}$ and $\mathbb{R}^{d+1}$ respectively.
We put,
$\frac{1}{2}\varphi^2 := a_{00}, \quad a_{ij} := \frac{1}{2}(\sigma^{(1)}\sigma^{(1)\,*})_{ij},\, i,\, j = 1,\, ...,\, d, \quad$ and $\quad \sigma = \begin{pmatrix} \varphi & 0 \\ 0 & \sigma^{(1)} \end{pmatrix}$.
One has $\sigma \in \mathbb{R}^{(d+1)\times k}$ with

$$\begin{cases} \sigma_{00} = \varphi, \\ \sigma_{0j} = 0,\; j = 1,\, ...,\, k-1 \\ \sigma_{i0} = 0,\; i = 1,\, ...,\, d \\ \sigma_{ij} = \sigma_{ij}^{(1)},\; i = 1,\, ...,\, d,\; j = 1,\, ...,\, k-1 \end{cases}$$

We denote, $X^\varepsilon := (X^{1,\,\varepsilon},\, X^{2,\,\varepsilon}), \quad b = (0,\, b^{(1)})^*, \quad$ and $\quad B = (W,\, \widetilde{W})$
The PDE (1.2) is then connected to the Markovian FBSDEs,

$$\begin{cases} X_s^\varepsilon = x + \int_0^s b(\frac{X_u^{1,\,\varepsilon}}{\varepsilon},\, X_u^{2,\,\varepsilon})du + \int_0^s \sigma(\frac{X_u^{1,\,\varepsilon}}{\varepsilon},\, X_u^{2,\,\varepsilon})dB_u, \\ \\ Y_s^\varepsilon = H(X_t^\varepsilon) + \int_s^t f(\frac{X_u^{1,\,\varepsilon}}{\varepsilon},\, X_u^{2,\,\varepsilon},\, Y_u^\varepsilon)du - \int_s^t Z_u^\varepsilon\, dM_u^{X^\varepsilon},\, \forall\, s \in [0,\, t] \end{cases} \tag{1.3}$$

where $x = (x^1, x^2)$ and M^{X^ε} is a martingale part of the process X^ε.

It is well known that (under some conditions) the representation $v^\varepsilon(t,\, x) = Y_0^\varepsilon$ holds.

The aim of the present paper is:

1) to show that the sequence of process $(X_t^\varepsilon, Y_t^\varepsilon, \int_s^t Z_u^\varepsilon\, dM_u^{X^\varepsilon})_{0\le s\le t}$ converges in law to the process $(X_t, Y_t, \int_s^t Z_u\, dM_u^X)_{0\le s\le t}$ which is the unique solution to the FBSDE,

$$\begin{cases} X_s = x + \int_0^s \overline{b}(X_u)du + \int_0^s \overline{\sigma}(X_u)dB_u,\; 0 \le s \le t. \\ \\ Y_s = H(X_t) + \int_s^t \overline{f}(X_u,\, Y_u)du - \int_s^t Z_u dM_u^X, 0 \le s \le t \end{cases} \tag{1.4}$$

where $\overline{\sigma}$, $\overline{b}$ and $\overline{f}$ are respectively the average of σ, b and f.

2) As a consequence, we establish that v^ε tends towards v, which solves the following averaged equation in the L^p-viscosity sense.

$$\begin{cases} \frac{\partial v}{\partial s}(s,\, x_1,\, x_2) = \overline{L}(x_1,\, x_2)v(s, x_1,\, x_2) + \overline{f}(x_1,\, x_2,\, v(s,\, x_1,\, x_2)) \quad 0 < s \le t \\ \\ v(0, x_1,\, x_2) = H(x_1,\, x_2) \end{cases} \tag{1.5}$$

where $\overline{L}(x_1, x_2) = \sum_{i,j} \overline{a}_{ij}(x_1, x_2)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \overline{b}_i(x_1, x_2)\frac{\partial}{\partial x_i}$, is the averaged operator.

The method used to derive the averaged BSDE is based on weak convergence in the **S**-topology and is close to that used in [20]. In our framework, we show that the limit FBSDE (1.4) has a unique solution. However, due to the discontinuity of the coefficients, the classical viscosity solution is not defined for the averaged PDE (1.5). We then use the notion of "L^p-viscosity solution". We use BSDEs techniques to establish the existence of L^p-viscosity solution for the averaged PDE. The notion of L^p-viscosity solution has been introduced by Cafarelli *et al.* in [6] to study fully nonlinear PDEs with measurable coefficients. However, even if the notion of L^p-viscosity solution is available for PDEs with merely measurable coefficients, one require continuity property for such solutions. In our case, the lack of L^2-continuity property for the flow $X^x := (X^{1,x}, X^{2,x})$ transfer the difficulty to the backward one and hence we cannot prove the L^2-continuity of the process Y. To overcome this difficulty, we establish weak continuity for the flow $x \mapsto (X^{1,x}, X^{2,x})$ and use the fact that Y_0^x is deterministic, to derive the continuity property for Y_0^x.

The paper is organized as follows: In section 2, we make some notations and assumptions. Our main results are stated in section 3. Section 4 and 5 are devoted to the proofs.

2 Notations and assumptions

2.1 Notations

For a given function $g(x_1, x_2)$, we define

$$g^+(x_2) := \lim_{x_1 \to +\infty} \frac{1}{x_1}\int_0^{x_1} g(t, x_2)dt$$

$$g^-(x_2) := \lim_{x_1 \to -\infty} \frac{1}{x_1}\int_0^{x_1} g(t, x_2)dt$$

The limit in Ċesaro of g is defined by,

$$g^{\pm}(x_1, x_2) := g^+(x_2)1_{\{x_1>0\}} + g^-(x_2)1_{\{x_1\leq 0\}}$$

Let $\rho(x_1, x_2) := a_{00}(x_1, x_2)^{-1}(= [\frac{1}{2}\varphi^2(x_1, x_2)]^{-1})$ and denote by $\overline{b}(x_1, x_2)$, $\overline{a}(x_1, x_2)$ and $\overline{f}(x_1, x_2, y)$, the averaged coefficients defined as follows,

$$\overline{b}_i(x_1, x_2) = \frac{(\rho b_i)^{\pm}(x_1, x_2)}{\rho^{\pm}(x_1, x_2)}, \qquad i = 1, ..., d$$

$$\overline{a}_{ij}(x_1, x_2) = \frac{(\rho a_{ij})^{\pm}(x_1, x_2)}{\rho^{\pm}(x_1, x_2)}, \qquad i, j = 0, 1, ..., d$$

$$\overline{f}(x_1, x_2, y) = \frac{(\rho f)^{\pm}(x_1, x_2, y)}{\rho^{\pm}(x_1, x_2)},$$

It's worth noting that $\overline{b}$, $\overline{a}$ and $\overline{f}$ may be discontinuous at $x_1 = 0$.

2.2 Assumptions.

We consider the following conditions,

(A1) The function $b^{(1)}$, $\sigma^{(1)}$, φ are uniformly Lipschitz in the variables $(x_1,\, x_2)$,

(A2) for each x_1, their derivative in x_2 up to and including second order derivatives are bounded continuous functions of x_2.

(A3) $a := (\sigma^{(1)}\sigma^{(1)\,*})$ is uniformly elliptic, $i.e$: $\exists \Lambda > 0; \quad \forall x, \xi \in \mathbb{R}^d, \quad \xi^* a(x)\xi \geq \Lambda|\xi|^2$.
Moreover, there exist positive constants C_1, C_2, C_3 such that

$$\begin{cases} (i)\ C_1 \leq a_{00}(x_1,\, x_2) \leq C_2 \\ \\ (ii)\ |a(x_1,\, x_2)| + |b(x_1,\, x_2)|^2 \leq C_3(1 + |x_2|^2) \end{cases}$$

(B1) Let $D_{x_2}u$ and $D^2_{x_2}u$ denote respectively the gradient vector and the matrix of second derivatives of u with respect to x_2. The following limits are uniform in x_2,

$$\frac{1}{x_1}\int_0^{x_1} \rho(t,\, x_2)dt \longrightarrow \rho^{\pm}(x_2) \qquad \text{as} \quad x_1 \to \pm\infty$$

$$\frac{1}{x_1}\int_0^{x_1} D_{x_2}\rho(t,\, x_2)dt \longrightarrow D_{x_2}\rho^{\pm}(x_2) \qquad \text{as} \quad x_1 \to \pm\infty$$

$$\frac{1}{x_1}\int_0^{x_1} D^2_{x_2}\rho(t,\, x_2)dt \longrightarrow D^2_{x_2}\rho^{\pm}(x_2) \qquad \text{as} \quad x_1 \to \pm\infty$$

(B2) For every i and j, the coefficients ρb_i, $D_{x_2}(\rho b_i)$, $D^2_{x_2}(\rho b_i)$, ρa_{ij}, $D_{x_2}(\rho a_{ij})$, $D^2_{x_2}(\rho a_{ij})$ have limits in Cèsaro sense.

(B3) For every function $k \in \{\rho,\, \rho b_i,\, D_{x_2}(\rho b_i),\, D^2_{x_2}(\rho b_i),\, \rho a_{ij},\, D_{x_2}(\rho a_{ij}),\, D^2_{x_2}(\rho a_{ij})\}$, there exists a bounded function α such that

$$\begin{cases} \frac{1}{x_1}\int_0^{x_1} k(t,\, x_2)dt - k^{\pm}(x_1,\, x_2) = (1 + |x_2|^2)\alpha(x_1,\, x_2), \\ \\ \lim\limits_{|x_1| \longrightarrow \infty} \sup\limits_{x_2 \in \mathbb{R}^d} |\alpha(x_1,\, x_2)| = 0. \end{cases} \tag{2.1}$$

(C1)

(i) For every $(x_1, x_2) \in \mathbb{R}^{d+1}$, $f(x_1, x_2, \cdot) \in C^2_b(\mathbb{R})$, the bounds being uniform in (x_1, x_2).

(ii) H is continuous and bounded.

(C2) ρf has a limit in Cèsaro sense and there exists a bounded measurable function β such that

$$\begin{cases} \frac{1}{x_1}\int_0^{x_2} \rho(t,\, x_2) f(t,\, x_2,\, y)dt - (\rho f)^{\pm}(x_1,\, x_2,\, y) = (1 + |x_2|^2 + |y|^2)\beta(x_1,\, x_2,\, y) \\ \\ \lim\limits_{|x_1| \to \infty} \sup\limits_{(x_2,\, y) \in \mathbb{R}^d \times \mathbb{R}} |\beta(x_1,\, x_2,\, y)| = 0, \end{cases} \tag{2.2}$$

(C3) For each x_1, ρf has derivatives up to second order in (x_2, y) and these derivatives are bounded and satisfy (C2).

Throughout the paper, **(A)** stands for conditions (A1), (A2), (A3); **(B)** for conditions (B1), (B2), (B3) and **(C)** for (C1), (C2), (C3).

3 The main result.

Consider the equation

$$X_s^{t,x} = x + \int_t^s \overline{b}(X_u^{t,x})du + \int_t^s \overline{\sigma}(X_u^{t,x})dB_u,\ t \leq s \leq T \tag{3.1}$$

Assume that **(A), (B)** hold. Then,

• From Khasminskii and Krylov [13] we have: *the process* $X^\varepsilon := (X^{1,\varepsilon},\, X^{2,\varepsilon})$ *converges in distribution to the process* $X := (X^1,\, X^2)$.

and

• From Krylov [16] we have: The limit $X = (X^1,\, X^2)$ is a unique weak solution to SDE (3.1).

We now define the L^p-viscosity solution. This notion has been introduced by Caffarelli *et al.* in [6] to study PDEs with measurable coefficients. Wide presentation of this topic can be found in [6, 7].

Let $g : [0,\, T] \times \mathbb{R}^{d+1} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function and

$\overline{L}(x_1,\, x_2) := \sum_{i,j} \overline{a}_{ij}(x_1,\, x_2)\dfrac{\partial^2}{\partial x_i \partial x_j} + \sum_i \overline{b}_i(x_1,\, x_2)\dfrac{\partial}{\partial x_i}$ be the operator associated to the solution to SDE (3.1).

We consider the parabolic equation,

$$\begin{cases} \frac{\partial v}{\partial s}(s,\, x_1,\, x_2) + \overline{L}(x_1,\, x_2)v(s, x_1,\, x_2) + g(s,\, x_1,\, x_2,\, v(s,\, x_1,\, x_2)) = 0,\ t \leq s < T \\ \\ v(T, x_1,\, x_2) = H(x_1,\, x_2). \end{cases} \tag{3.2}$$

Definition 3.1. Let p be an integer such that $p > d + 2$.

(a)- A function $v \in \mathcal{C}\left([0,\, T] \times \mathbb{R}^{d+1},\, \mathbb{R}\right)$ is a L^p-viscosity sub-solution of the PDE (3.2), if for every $x \in \mathbb{R}^{d+1}$, $v(T,\, x) \leq H(x)$ and for every $\varphi \in W^{1,2}_{p,\,loc}\left([0,\, T] \times \mathbb{R}^{d+1},\, \mathbb{R}\right)$ and $(\widehat{t},\, \widehat{x}) \in [0,\, T] \times \mathbb{R}^{d+1}$ at which $v - \varphi$ has a local maximum, one has

$$ess \lim_{(s,x)\to(\widehat{t},\widehat{x})} \inf \left\{ -\frac{\partial \varphi}{\partial s}(s,\, x_1,\, x_2) - G(s,\, x,\, \varphi(s,\, x)) \right\} \leq 0.$$

(b)- A function $v \in \mathcal{C}\left([0,\, T] \times \mathbb{R}^{d+1},\, \mathbb{R}\right)$ is a L^p-viscosity super-solution of the PDE (3.2), if for every $x \in \mathbb{R}^{d+1}$, $v(T,\, x) \geq H(x)$ and for every $\varphi \in W^{1,2}_{p,\,loc}\left([0,\, T] \times \mathbb{R}^{d+1},\, \mathbb{R}\right)$ and $(\widehat{t},\, \widehat{x}) \in [0,\, T] \times \mathbb{R}^{d+1}$ at which $v - \varphi$ has a local minimum, one has

$$ess \lim_{(s,x)\to(\widehat{t},\widehat{x})} \sup \left\{ -\frac{\partial \varphi}{\partial s}(s,\, x_1,\, x_2) - G(s,\, x,\, \varphi(s,\, x)) \right\} \geq 0.$$

Here, $G(s,\, x,\, \varphi(s,\, x)) = \overline{L}(x_1,\, x_2)\varphi(s, x_1,\, x_2) + g(s,\, x_1,\, x_2,\, v(s,\, x_1,\, x_2))$ is assumed to be merely measurable on the variable $x = (x_1,\, x_2)$.

(c)- A function $v \in \mathcal{C}\left([0,\, T] \times \mathrm{I\!R}^{d+1},\, \mathrm{I\!R}\right)$ is a L^p-viscosity solution if it is both a L^p-viscosity sub-solution and super-solution.

Remark 3.2. The definition (a) means that for every $\varepsilon > 0$, $r > 0$, there exists a set $A \subset B_r(\widehat{t},\, \widehat{x})$ of positive measure such that, $\forall\, (s,\, x) \in A$,

$$-\frac{\partial \varphi}{\partial s}(s,\, x_1,\, x_2) - \overline{L}(x_1,\, x_2)\varphi(s, x_1,\, x_2) - g(s,\, x_1,\, x_2,\, v(s,\, x_1,\, x_2)) \leq \varepsilon.$$

The main result is,

Theorem 3.3. *Assume (A), (B), (C). Then, the sequence of process* $(X_t^\varepsilon, Y_t^\varepsilon, \int_s^t Z_u^\varepsilon\, dM_u^{X^\varepsilon})$ *converges in law to the process* $(X_t, Y_t, \int_s^t Z_u\, dM_u^X)$ *which is the unique solution to FBSDE (1.4).*

Theorem 3.4. *Assume (A), (B), (C). For* $\varepsilon > 0$, *let* v^ε *be the unique solution to the problem (1.2). Let* $(Y_s^{(t,x)})_s$ *be the unique solution of BSDE (1.4).*
Then,

(i) Equation (1.5) has a unique L^p*-viscosity solution* v *such that* $v(t,x) = Y_0^{(t,x)}$.

(ii) for every (t,x), $v^\varepsilon(t,x)$ *converges to* $v(t,x)$.

Remark 3.5. **1)** The conclusion of Theorem 3.3 remains valid if we take the forward component of our FBSDE (1.3) as that of [13] and replace the Brownian motion B by W. The dimensions and the assumptions should be accordingly rearranged. For instance, in place equation (1.3), we consider the FBSDE,

$$\begin{cases} X_s^\varepsilon = x + \int_0^s b(\frac{X_u^{1,\,\varepsilon}}{\varepsilon},\, X_u^{2,\,\varepsilon})du + \int_0^s \sigma(\frac{X_u^{1,\,\varepsilon}}{\varepsilon},\, X_u^{2,\,\varepsilon})dW_u, \\ \\ Y_s^\varepsilon = H(X_t^\varepsilon) + \int_s^t f(\frac{X_u^{1,\,\varepsilon}}{\varepsilon},\, X_u^{2,\,\varepsilon},\, Y_u^\varepsilon)du - \int_s^t Z_u^\varepsilon\, dM_u^{X^\varepsilon},\, \forall\, s \in [0,\, t] \end{cases} \tag{3.3}$$

where $x = (x_1, x_2)$ and

$$\begin{cases} X_t^{1,\,\varepsilon} = x_1 + \int_0^t \varphi(\frac{X_s^{1,\,\varepsilon}}{\varepsilon},\, X_s^{2,\,\varepsilon})dW_s \\ \\ X_t^{2,\,\varepsilon} = x_2 + \int_0^t b^{(1)}(\frac{X_s^{1,\,\varepsilon}}{\varepsilon},\, X_s^{2,\,\varepsilon})ds + \int_0^t \sigma^{(1)}(\frac{X_s^{1,\,\varepsilon}}{\varepsilon},\, X_s^{2,\,\varepsilon})dW_s \end{cases} \tag{3.4}$$

In this case: φ is a $\mathrm{I\!R}^k-$valued function, W is an $\mathrm{I\!R}^k-$Brownian motion and $\sigma^{(1)} = (\sigma_{ij}^{(1)})$ is a $d \times k$ matrix. The nondegeneracy condition A3)-i) imposed on φ should be replaced by, $C_1 \leq \sum_{i=1}^k \varphi_i(x_1,\, x_2) \leq C_2$. The infinitesimal generator $\mathcal{L}$ associated to the diffusion component will be more complicated since it takes account on other (mixed) second order derivatives.

2) If in Theorem 3.3, we replace the initial condition of the forward component in equation (1.3) by $(\varepsilon x_1, x_2)$, then we obtain the same limit with initial condition $(0, x_2)$ instead of (x_1, x_2).

4 Proof of Theorem 3.3.

4.1 Tightness and convergence for the BSDE.

Proposition 4.1. *There exists a positive constant C which does not depend on ε such that*

$$\sup_{\varepsilon} I\!E\left(\sup_{0\le s\le t}|Y_s^\varepsilon|^2+\int_0^t|Z_s^\varepsilon|^2\,d\langle M^{X^\varepsilon}\rangle_s\right)\le C.$$

Proof. Throughout this proof, K, C are positive constants which depends only on $(s,\,t)$ and may change from line to line. It is easy to check that for all $k\ge 1$,

$$\sup_{\varepsilon}\mathbb{E}\left(\sup_{0\le s\le t}\left[|X_s^{1,\varepsilon}|^{2k}+|X_s^{2,\varepsilon}|^{2k}\right]\right)<+\infty. \tag{4.1}$$

Using Itô's formula, we get:

$$\begin{aligned}|Y_s^\varepsilon|^2+\int_s^t|Z_u^\varepsilon|^2 d\langle\, M^{X^\varepsilon}\,\rangle_u &\le |H(X_t^\varepsilon)|^2+K\int_s^t|Y_u^\varepsilon|^2du+\int_s^t|f(\overline{X}_u^{1,\varepsilon},\,X_u^{2,\varepsilon},\,0)|^2du\\ &- \;2\int_s^t\langle Y_u^\varepsilon,\,Z_u^\varepsilon dM_s^{X^\varepsilon}\rangle.\end{aligned}$$

Passing to expectation, it is then follows by using Gronwall's lemma that, there exists a positive constant C which does not depend on ε such that,

$$\mathbb{E}\left(|Y_s^\varepsilon|^2\right)\le C\mathbb{E}\left(|H(X_t^\varepsilon)|^2+\int_0^t|f(\overline{X}_u^{1,\varepsilon},\,X_u^{2,\varepsilon},\,0)|^2du\right),\ \forall s\in[0,\,t]$$

We deduce that

$$\mathbb{E}\left(\int_s^t|Z_u^\varepsilon|^2 d\langle\, M^{X^\varepsilon}\,\rangle_u\right)\le C\mathbb{E}\left(|H(X_t^\varepsilon)|^2+\int_0^t|f(\overline{X}_u^{1,\varepsilon},\,X_u^{2,\varepsilon},\,0)|^2du\right) \tag{4.2}$$

Combining (4.2) and the Burkhölder-Davis-Gundy inequality, we get

$$\mathbb{E}\left(\sup_{0\le s\le t}|Y_t^\varepsilon|^2+\frac{1}{2}\int_0^t|Z_u^\varepsilon|^2 d\langle\, M^{X^\varepsilon}\,\rangle_u\right)\le C\mathbb{E}\left(|H(X_t^\varepsilon)|^2+\int_0^t|f(\overline{X}_u^{1,\varepsilon},\,X_u^{2,\varepsilon},\,0)|^2du\right)$$

In view of condition $(C1-iii)$ and (4.1), the proof is complete. ∎

Proposition 4.2. *For $\varepsilon>0$, let Y^ε be the process defined by equation 1.3 and M^ε) be its martingale part. The sequence $(Y^\varepsilon,\,M^\varepsilon)_{\varepsilon>0}$ is tight on the space $\mathcal{D}([0,\,t],\,I\!R)\times\mathcal{D}([0,\,t],\,I\!R)$ endowed with the* **S***-topology.*

Proof. Since M^ε is a martingale, then by [18] or [12], the Meyer-Zheng tightness criteria is fulfilled whenever

$$\sup_{\varepsilon}\left(CV(Y^\varepsilon)+\mathbb{E}\left(\sup_{0\le s\le t}|Y_t^\varepsilon|+|M_t^\varepsilon|\right)\right)<+\infty. \tag{4.3}$$

where the conditional variation CV is defined in appendix A.
By [22], the conditional expectation $CV(Y^\varepsilon)$ satisfies

$$CV(Y^\varepsilon)\le K\mathbb{E}\left(\int_0^t|f(\overline{X}_s^{1,\varepsilon},\,X_s^{2,\varepsilon},\,Y_s^\varepsilon)|^2ds\right),$$

where K is a constant which only depends on t.
Combining condition (*C1*) and Proposition 4.1, we deduce (4.3). ∎

Proposition 4.3. *There exists $(Y,\, M)$ and a countable subset D of $[0,\, t]$ such that along a subsequence of ε,*

(i) $(Y^\varepsilon, M^\varepsilon) \overset{law}{\Longrightarrow} (Y, M)$ *on* $\mathcal{D}\left([0,\, t],\, I\!R\right) \times \mathcal{D}\left([0,\, t],\, I\!R\right)$ *endowed with the* **S***-topology.*

(ii) $(Y^\varepsilon,\, M^\varepsilon) \longrightarrow (Y,\, M)$ *in finite-distribution on* D^c*.*

(iii) $(X^{1,\varepsilon}, X^{2,\varepsilon}, Y^\varepsilon) \Rightarrow (X^1, X^2, Y)$ *, in the sense of weak convergence in* $C(I\!R_+ \times I\!R^{d+1}) \times D(I\!R_+ \times I\!R)$*, equipped with the product of the uniform convergence on compact sets and the* **S** *topologies.*

Proof. (i)- From Proposition 4.2, the family $\left(Y^\varepsilon,\, M^\varepsilon\right)_\varepsilon$ is tight on $\mathcal{D}\left([0,\, t],\, \mathbb{R}\right)\times\mathcal{D}\left([0,\, t],\, \mathbb{R}\right)$ endowed with the **S**-topology. Hence, along a subsequence (still denoted by ε), $\left(Y^\varepsilon,\, M^\varepsilon\right)_\varepsilon$ converges in law on $\mathcal{D}\left([0,\, t],\, \mathbb{R}\right) \times \mathcal{D}\left([0,\, t],\, \mathbb{R}\right)$ towards a càd-làg process $\left(\overline{Y},\, \overline{M}\right)$.

(ii)- Thanks to Theorem 3.1 in Jakubowski [12], there exists a countable set D such that along a subsequence the convergence in law holds. Moreover, the convergence in finite-distribution holds on D^c. ■

4.2 Identification of the limit finite variation process.

Proposition 4.4. *Let (Y, M), the limit process defined in Proposition 4.3. Then,*
(i) For every $s \in [0,\, t] - \mathsf{D}$*,*

$$\begin{cases} Y_s = H(X_t) + \int_s^t \overline{f}(X_u^1,\ X_u^2, Y)du - (M_t - M_s), \\ \\ I\!E\left(\sup_{0\le s\le t} |Y_s|^2 + |X_s^1|^2 + |X_s^2|^2\right) \le C. \end{cases} \tag{4.4}$$

(ii) M *is a* $\mathcal{F}_s$*-martingale, where* $\mathcal{F}_s := \sigma\{X_u,\, Y_u, \quad 0 \le u \le s\}$ *augmented with the* $I\!P$*-null sets.*

To prove this proposition, we need the following lemmas.

Lemma 4.5. *Assume* **(A), (B)***, (C2) and (C3). For* $y \in I\!R$*, let* $V^\varepsilon(x_1,\, x_2, y)$ *denote the solution of the following equation:*

$$\begin{cases} a_{00}(\frac{x_1}{\varepsilon},\, x_2)D_{x_1}^2 u(x_1,\, x_2, y) = f(\frac{x_1}{\varepsilon},\, x_2,\, y) - \overline{f}(x_1,\, x_2,\, y) \\ \\ u(0,\, x_2) = D_{x_1}u(0,\, x_2) = 0. \end{cases} \tag{4.5}$$

Then,
(i) $\quad D_{x_1}V^\varepsilon(x_1,\, x_2, y) = x_1(1 + |x_2|^2 + |y|^2)\beta(\dfrac{x_1}{\varepsilon},\, x_2,\, y) - x_1(1 + |x_2|^2)m(\dfrac{x_1}{\varepsilon}, x_2,\, y),$

(ii) for any $K^\varepsilon(x_1,\, x_2, y) \in \left\{V^\varepsilon,\, D_{x_2}V^\varepsilon,\, D_{x_2}^2V^\varepsilon,\, D_{x_1}D_{x_2}V^\varepsilon, D_yV^\varepsilon, D_y^2V^\varepsilon, D_{x_1}D_yV^\varepsilon, D_{x_2}D_yV^\varepsilon\right\}$ *it holds,*

$$K^\varepsilon(x_1,\, x_2, y) = x_1^2(1 + |x_2|^2 + |y|^2)\beta(\frac{x_1}{\varepsilon},\, x_2,\, y) + x_1^2(1 + |x_2|^2)m(\frac{x_1}{\varepsilon},\, x_2,\, y)$$

where $m(\frac{x_1}{\varepsilon},\, x_2,\, y) := \frac{(\rho f)^\pm(x_1, x_2, y)}{\rho^\pm(x_1, x_2)}\alpha(\frac{x_1}{\varepsilon},\, x_2)$ *and* $\alpha(x_1,\, x_2)$*,* $\beta(x_1,\, x_2,\, y)$ *are various bounded functions which satisfy property (2.1)and (2.2) respectively.*

Proof. We will adapt the idea of [13] to our situation. For a fixed y, we set

$$F(\frac{x_1}{\varepsilon},\, x_2, y) := \frac{1}{x_1}\int_0^{x_1} \rho(\frac{t}{\varepsilon},\, x_2) g(\frac{t}{\varepsilon},\, x_2,\, y)dt$$

where $g(\frac{x_1}{\varepsilon},\, x_2,\, y) := f(\frac{x_1}{\varepsilon},\, x_2,\, y) - \overline{f}(x_1,\, x_2,\, y)$.
For $x_1 > 0$, we have

$$\begin{aligned}
F(\frac{x_1}{\varepsilon},\, x_2, y) &= \frac{1}{x_1}\int_0^{x_1} \rho(\frac{t}{\varepsilon},\, x_2) f(\frac{t}{\varepsilon},\, x_2,\, y)dt - (\rho f)^+(x_2,\, y) \\
&+ (\rho f)^+(x_2,\, y) - \frac{(\rho f)^+(x_2,\, y)}{\rho^+(x_2)}\frac{1}{x_1}\int_0^{x_1} \rho(\frac{t}{\varepsilon},\, x_2)dt \\
&= (1+|x_2|^2+|y|^2)\beta_1(\frac{x_1}{\varepsilon},\, x_2,\, y) \\
&+ (\rho f)^+(x_2,\, y)\left[1 - \frac{1}{\rho^+(x_2)x_1}\int_0^{x_1} \rho(\frac{t}{\varepsilon},\, x_2)dt\right] \\
&= (1+|x_2|^2+|y|^2)\beta_1(\frac{x_1}{\varepsilon},\, x_2,\, y) - (1+|x_2|^2)\frac{(\rho f)^+(x_2,\, y)}{\rho^+(x_2)}\alpha_1(\frac{x_1}{\varepsilon},\, x_2)
\end{aligned}$$

Since, $D_{x_1}V^\varepsilon(x_1,\, x_2, y) = x_1 F(\frac{x_1}{\varepsilon},\, x_2, y)$, we derive the result for $D_{x_1}V^\varepsilon(x_1,\, x_2, y)$. Further, by integrating, we get

$$\begin{aligned}
V^\varepsilon(x_1,\, x_2, y) &= x_1^2(1+|x_2|^2+|y|^2)\left((\frac{\varepsilon}{x_1})^2\int_0^{\frac{x_1}{\varepsilon}} t\beta_1(t,\, x_2,\, y)dt\right) \\
&- (1+|x_2|^2)\frac{(\rho f)^+(x_2,\, y)}{\rho^+(x_2)}\left((\frac{\varepsilon}{x_1})^2\int_0^{\frac{x_1}{\varepsilon}} t\alpha_1(t,\, x_2)dt\right)
\end{aligned}$$

Clearly, $\beta(\frac{x_1}{\varepsilon},\, x_2,\, y) := (\frac{\varepsilon}{x_1})^2\int_0^{\frac{x_1}{\varepsilon}} t\beta_1(t,\, x_2,\, y)dt,\ \alpha(\frac{x_1}{\varepsilon},\, x_2) = (\frac{\varepsilon}{x_1})^2\int_0^{\frac{x_1}{\varepsilon}} t\alpha(t,\, x_2)dt$ satisfy (2.1) and (2.2) respectively. The same argument can be used in the case $x_1 < 0$. The result for $D_{x_2}V^\varepsilon(x_1,\, x_2, y)$, $D^2_{x_2}V^\varepsilon(x_1, x_2, y)$ and $D_{x_1}D_{x_2}V^\varepsilon(x_1, x_2, y)$ can obtained by using similar arguments. ■

Lemma 4.6. $\sup_{0\le s\le t}\left|\int_0^s \left(f(\frac{X_u^{1,\varepsilon}}{\varepsilon},\, X_u^{2,\varepsilon},\, Y_u^\varepsilon) - \overline{f}(X_u^{1,\varepsilon},\, X_u^{2,\varepsilon},\, Y_u^\varepsilon)\right)du\right|$ *tends to zero in probability as* $\varepsilon \longrightarrow 0$.

Proof. Let V^ε denote the solution of equation (4.5). Note that V^ε has first and second derivatives in $(x_1,\, x_2, y)$ which are possibly discontinuous only at $x_1 = 0$. Then, as in [13], since φ is non degenerate, we can use Itô-Krylov formula to get

$$
\begin{aligned}
V^\varepsilon(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon) &= V^\varepsilon(x_1, x_2, Y_0^\varepsilon) + \int_0^s \left[f(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon) - \overline{f}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)\right]du \\
+ & \int_0^s a_{ij}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon})\frac{\partial^2 V^\varepsilon}{\partial x_{2i}\partial x_{2j}}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)du + \int_0^s b_j^{(1)}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon})\frac{\partial V^\varepsilon}{\partial x_{2j}}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)du \\
+ & \int_0^s [\frac{\partial V^\varepsilon}{\partial x_1}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)\varphi(X_u^{1,\varepsilon}, X_u^{2,\varepsilon})dW_u + \frac{\partial V^\varepsilon}{\partial x_2}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)\sigma^{(1)}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon})d\widetilde{W}_u] \\
- & \int_0^s \frac{\partial V^\varepsilon}{\partial y}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)f(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)du + \int_0^s \frac{\partial V^\varepsilon}{\partial y}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)Z_u^\varepsilon\sigma(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon})dB_u \\
+ & \frac{1}{2}\int_0^s \frac{\partial^2 V^\varepsilon}{\partial y\partial y}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)Z_u^\varepsilon\sigma\sigma^*(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon})(Z_u^\varepsilon)^*du \qquad (4.6) \\
+ & \frac{1}{2}\int_0^s \frac{\partial^2 V^\varepsilon}{\partial y\partial x}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)\sigma\sigma^*(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon})(Z_u^\varepsilon)^*du.
\end{aligned}
$$

In view of Lemma 4.5, it is obvious to see that $V^\varepsilon(x_1, x_2, Y_0^\varepsilon)$ tends to zero. Once again, from Lemma 4.5, we have

$$
\begin{aligned}
\left|V^\varepsilon(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)\right| \quad \leq \quad & \varepsilon\left[(1+|X_s^{2,\varepsilon}|^2+|Y_s^\varepsilon|^2)|\beta(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)|\right] \\
+ \; & \varepsilon\left[(1+|X_s^{2,\varepsilon}|^2)\frac{(\rho f)^\pm(X_s^{2,\varepsilon}, Y_s^\varepsilon)}{\rho^\pm(X_s^{2,\varepsilon})}|\alpha(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon})|\right] \\
+ \; & 1_{\{|X_s^{1,\varepsilon}|\geq\sqrt{\varepsilon}\}}|X_s^{1,\varepsilon}|^2\left[(1+|X_s^{2,\varepsilon}|^2+|Y_s^\varepsilon|^2)|\beta(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon, Z_s^{\varepsilon,n})\right] \\
+ \; & 1_{\{|X_s^{1,\varepsilon}|\geq\sqrt{\varepsilon}\}}|X_s^{1,\varepsilon}|^2\left[(1+|X_s^{2,\varepsilon}|^2)\frac{(\rho f)^\pm(X_s^{2,\varepsilon}, Y_s^\varepsilon)}{\rho^\pm(X_s^{2,\varepsilon})}|\alpha(\frac{X_s^{1,\varepsilon}}{\varepsilon}, X_s^{2,\varepsilon})|\right]
\end{aligned}
$$

From the estimates of the processes $X_s^{1,\varepsilon}$, $X_s^{2,\varepsilon}$, Y_s^ε and the fact that $(\rho f)^\pm$ satisfies conditions (**C**), we deduce that

$$
\mathbb{E}\left(\sup_{0\leq s\leq t}|V^{\varepsilon,n}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)|\right) \leq K\left(\varepsilon + \sup_{|x_1|\geq\sqrt{\varepsilon}}\sup_{(x_2,y)}|\beta(\frac{x^1}{\varepsilon}, x^2, y)| + \sup_{|x_1|\geq\sqrt{\varepsilon}}\sup_{x_2}|\alpha(\frac{x^1}{\varepsilon}, x^2)|\right)
$$

Then, since α and β satisfy (2.1) and (2.2) respectively, the right hand side of the previous inequality tends to zero as $\varepsilon \longrightarrow 0$. Similarly, one can show that

$$
\begin{aligned}
+ & \int_0^s a_{ij}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon})\frac{\partial^2 V^\varepsilon}{\partial x_{2i}\partial x_{2j}}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)du + \int_0^s b_j^{(1)}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon})\frac{\partial V^\varepsilon}{\partial x_{2j}}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)du \\
+ & \int_0^s [\frac{\partial V^\varepsilon}{\partial x_1}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)\varphi(X_u^{1,\varepsilon}, X_u^{2,\varepsilon})dW_u + \frac{\partial V^\varepsilon}{\partial x_2}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)\sigma^{(1)}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon})d\widetilde{W}_u] \\
- & \int_0^s \frac{\partial V^\varepsilon}{\partial y}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)f(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)du + \int_0^s \frac{\partial V^\varepsilon}{\partial y}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)Z_u^\varepsilon\sigma(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon})dB_u \\
+ & \frac{1}{2}\int_0^s \frac{\partial^2 V^\varepsilon}{\partial y\partial y}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)Z_u^\varepsilon\sigma\sigma^*(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon})(Z_u^\varepsilon)^*du \\
+ & \frac{1}{2}\int_0^s \frac{\partial^2 V^\varepsilon}{\partial y\partial x}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)\sigma\sigma^*(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon})(Z_u^\varepsilon)^*du
\end{aligned}
$$

converge to zero in probability. Let us give an explanation concerning the one but last term, which is the most delicate one.

$$\begin{aligned}&\left|\int_0^s \frac{\partial^2 V^\varepsilon}{\partial y \partial y}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon) Z_u^\varepsilon \sigma\sigma^*(\frac{X_u^{1,\varepsilon}}{\varepsilon}, X_u^{2,\varepsilon})(Z_u^\varepsilon)^* du\right| \\ &\leq C \sup_{0\leq u\leq s}(1+|X_u^{2,\varepsilon}|^2)\left|\frac{\partial^2 V^\varepsilon}{\partial y \partial y}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)\right| \int_0^s |Z_u^\varepsilon|^2 du\end{aligned}$$

Since $\{\int_0^s |Z_u^\varepsilon|^2 du,\ 0\leq s\leq t\}$ is the increasing process associated to a bounded martingale, so the $L^p(\mathbb{P})$ norm of $\int_0^t |Z_u^\varepsilon|^2 du$ is bounded, for all $p\geq 1$. Moreover $\sup_{0\leq u\leq t}(1+|X_u^{2,\varepsilon}|^2)$ is also bounded in all $L^p(\mathbb{P})$ spaces. Finally the same argument as above shows that

$$\sup_{0\leq u\leq s}\left|\frac{\partial^2 V^\varepsilon}{\partial y \partial y}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)\right| \to 0$$

in probability, as $\varepsilon \to 0$. ■

Lemma 4.7. *$\int_0^\cdot \overline{f}(X_u^{1,\varepsilon}, X_u^{2,\varepsilon}, Y_u^\varepsilon)du \overset{law}{\Longrightarrow} \int_0^\cdot \overline{f}(X_u^1, X_u^2, Y_u)du$ on $\mathcal{C}([0, t], I\!R)$ as $\varepsilon \longrightarrow 0$.*

For the proof of this Lemma, we need the following two results.

Lemma 4.8. *Let $X_s^1 := x_1 + \int_0^s \overline{\varphi}(X_u^1, X_u^2)dW_u$, $0\leq s\leq t$, and, assume (A2-i), (B1).*
For $\varepsilon > 0$, let $D_n^\varepsilon := \left\{s : s\in [0, t] \ /\ X_s^{1,\varepsilon} \in B(0, \frac{1}{n})\right\}$.
Define also $D_n := \left\{s : s\in [0, t] \ /\ X_s^{1} \in B(0, \frac{1}{n})\right\}$.
Then, there exists a constant $c > 0$ such that for each $n \geq 1$, $\varepsilon > 0$,

$$I\!E|D_n^\varepsilon| \leq \frac{c}{n} \qquad \textit{and} \qquad I\!E|D_n| \leq \frac{c}{n},$$

where $|.|$ denotes the Lebesgue measure.

Proof. Consider the sequence (Ψ_n) of functions defined as follows,

$$\Psi_n(x) = \begin{cases} -x/n - 1/2n^2 & if \quad x \leq -1/n \\ x^2/2 & if \quad -1/n \leq x \leq 1/n \\ x/n - 1/2n^2 & if \quad x \geq 1/n \end{cases}$$

We put, $\overline{\varphi} := \overline{a}_{00} := \rho(x_1, x_2)^{-1}$.

Using Itô's formula, we get

$$\Psi_n(X_s^1) = \Psi_n(X_0^1) + \int_0^s \Psi_n'(X_s^1)\overline{\varphi}(X_s^1, X_s^2)dW_s + \frac{1}{2}\int_0^s \Psi_n''(X_s^1)\overline{\varphi}^2(X_s^1, X_s^2)ds,\ s\in[0,t]$$

Since $\overline{\varphi}$ is lower bounded by C_1, taking the expectation, we get

$$\begin{aligned}C_1\mathbb{E}\int_0^t 1_{[-\frac{1}{n},\frac{1}{n}]}(X_s^1)ds &\leq \mathbb{E}\int_0^t \Psi_n''(X_s^1)\overline{\varphi}^2(X_s^1, X_s^2)ds \\ &= 2\mathbb{E}\left[\Psi_n(X_t^1) - \Psi_n(x_1)\right]\end{aligned}$$

It follows that $\mathbb{E}(|D_n|) \leq 2C_1^{-1}\mathbb{E}\left[\Psi_n(X_t^1) - \Psi_n(x_1)\right] \leq c/n$. The same argument, applies to D_n^ε, allows us to show the first estimate. ■

Lemma 4.9. *Consider a collection $\{Z^\varepsilon,\ \varepsilon > 0\}$ of real valued random variables, and a real valued random variable Z. Assume that for each $n \geq 1$, we have the decompositions*

$$\begin{aligned} Z^\varepsilon &= Z_n^{1,\varepsilon} + Z_n^{2,\varepsilon} \\ Z &= Z_n^1 + Z_n^2, \end{aligned}$$

such that for each fixed $n \geq 1$,

$$\begin{aligned} Z_n^{1,\varepsilon} &\Rightarrow Z_n^1 \\ I\!\!E|Z_n^{2,\varepsilon}| &\leq \frac{c}{\sqrt{n}} \\ I\!\!E|Z_n^2| &\leq \frac{c}{\sqrt{n}}. \end{aligned}$$

Then $Z^\varepsilon \Rightarrow Z$, as $\varepsilon \to 0$.

Proof. The above assumptions imply that the collection of random variables $\{Z^\varepsilon,\ \varepsilon > 0\}$ is tight. Hence the result will follow from the fact that

$$I\!\!E\Phi(Z^\varepsilon) \to I\!\!E\Phi(Z), \quad \text{as } \varepsilon \to 0$$

for all $\Phi \in C_b(I\!\!R)$ which is uniformly Lipschitz. Let Φ be such a function, and denote by K its Lipschitz constant. Then

$$\begin{aligned} |I\!\!E\Phi(Z^\varepsilon) - I\!\!E\Phi(Z)| &\leq I\!\!E|\Phi(Z^\varepsilon) - \Phi(Z_n^{1,\varepsilon})| + +|I\!\!E\Phi(Z_n^{1,\varepsilon}) - I\!\!E\Phi(Z_n^1)| + I\!\!E|\Phi(Z_n^1) - \Phi(Z)| \\ &\leq |I\!\!E\Phi(Z_n^{1,\varepsilon}) - I\!\!E\Phi(Z_n^1)| + 2K\frac{c}{\sqrt{n}}. \end{aligned}$$

Hence

$$\limsup_{\varepsilon \to 0} |I\!\!E\Phi(Z^\varepsilon) - I\!\!E\Phi(Z)| \leq 2K\frac{c}{\sqrt{n}},$$

for all $n \geq 1$. The result follows. ■

Proof of Lemma 4.7. For each $n \geq 1$, define a function $\theta_n \in C(I\!\!R, [0,1])$ such that $\theta_n(x) = 0$ for $|x| \leq 1/(2n)$, and $\theta_n(x) = 1$ for $|x| \geq 1/n$. We have

$$\begin{aligned} \int_0^t \overline{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)ds &= \int_0^t \overline{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)\theta_n(X_s^{1,\varepsilon})ds + \int_0^t \overline{f}(X_s^{1,\varepsilon}, X_s^{2,\varepsilon}, Y_s^\varepsilon)[1 - \theta_n(X_s^{1,\varepsilon})]ds \\ &= Z_n^{1,\varepsilon} + Z_n^{2,\varepsilon} \\ \int_0^t \overline{f}(X_s^1, X_s^2, Y_s)ds &= \int_0^t \overline{f}(X_s^1, X_s^2, Y_s)\theta_n(X_s^1)ds + \int_0^t \overline{f}(X_s^1, X_s^2, Y_s)[1 - \theta_n(X_s^1)]ds \\ &= Z_n^1 + Z_n^2 \end{aligned}$$

Note that the mapping

$$(x^1, x^2, y) \to \int_0^t \overline{f}(x_s^1, x_s^2, y_s)\theta_n(x_s^1)ds$$

is continuous from $C([0,t]) \times D([0,t])$ equipped with the product of the sup–norm and the **S** topologies into $I\!\!R$. Hence, from Proposition 4.3, $Z_n^{1,\varepsilon} \Rightarrow Z_n^1$ as $\varepsilon \to 0$, for each fixed $n \geq 1$.

Moreover, from Lemma 4.8, the linear growth property of $\overline{f}$, Proposition 4.1 and (4.1), we deduce that

$$E|Z_n^{2,\varepsilon}| \leq \frac{c}{\sqrt{n}}, \quad E|Z_n^2| \leq \frac{c}{\sqrt{n}}.$$

Lemma 4.7 now follows from Lemma 4.9. ■

Proof of Proposition 4.4 Passing to the limit in the backward component of the equation (1.3) and using Lemmas 4.6 and 4.7, we derive assertion (i).
Assertion (ii) can be proved by using the same arguments that developed in [21] section 6. ■

4.3 Identification of the limit martingale.

Proposition 4.10. *Let $(\bar{Y}_s, \bar{Z}_s,\ 0 \leq s \leq t)$ be the unique solution to BSDE $(H(X_t), \bar{f})$. Then, for every $s \in [0, t]$,*

$$I\!E|Y_s - \bar{Y}_s|^2 + I\!E\left([M - \int_0^\cdot \bar{Z}_u dM_u^X]_t - [M - \int_0^\cdot \bar{Z}_u dM_u^X]_s\right) = 0.$$

Proof. For every $s \in [0, t] - \mathsf{D}$, we have

$$\begin{cases} Y_s = H(X_t) + \int_s^t \overline{f}(X_u, Y_u)du - (M_t - M_s) \\ \bar{Y}_s = H(X_t) + \int_s^t \overline{f}(X_u, \bar{Y}_u)du - \int_s^t \bar{Z}_u dM_u^X \end{cases}$$

Arguing as in [21], we show that $\bar{M} := \int_s^\cdot \bar{Z}_u dM_u^X$ is a $\mathcal{F}_s$-martingale.
Since $\overline{f}$ satisfies condition $(C1)$, we get by Itô's formula, that

$$\mathbb{E}|Y_s - \overline{Y}_s|^2 + \mathbb{E}\left([M - \int_0^\cdot \bar{Z}_u dM_u^X]_t - [M - \int_0^\cdot \bar{Z}_u dM_u^X]_s\right) \leq C\mathbb{E}\int_s^t |Y_u - \overline{Y}_u|^2 du.$$

Therefore, Gronwall's lemma yields that $\mathbb{E}|Y_s - \overline{Y}_s|^2 = 0, \forall s \in [0, t] - \mathsf{D}$. But, $\bar{Y}$ is continuous, Y is càd-lag, and, D is countable. Hence, $Y_s = \overline{Y}_s$, $I\!P$-$a.s$, $\forall s \in [0, t]$.
Moreover, we deduce that, $\mathbb{E}\left([M - \int_0^\cdot \bar{Z}_u dM_u^X]_t - [M - \int_0^\cdot \bar{Z}_u dM_u^X]_s\right) = 0.$ ■

As a consequence of Proposition 4.10, we have

Corollary 4.11. $\left(Y^\varepsilon, \int_0^\cdot Z_u^\varepsilon dM_u^{X^\varepsilon}\right) \stackrel{law}{\Longrightarrow} \left(Y, \int_0^\cdot \bar{Z}_u dM_u^X\right).$

5 Proof of Theorem 3.4.

Since, under assumptions (A1) and (A2), the SDE (3.1) is weakly unique, the martingale problem associated to $X = (X^1, X^2)$ is well posed. We then have the following:

Proposition 5.1. *Assume that g satisfies $(C1)$. Then,*
(i) For a fixed positive number T, the BSDE

$$Y_s^{t,x} = H(X_T^{t,x}) + \int_s^T g(u, X_u^{t,x}, Y_u^{t,x})du - \int_s^T Z_u^{t,x} dM_u^{X^{t,x}}, t \leq s \leq T.$$

admits a unique solution $(Y_s^{t,x},\, Z_s^{t,x})_{t\le s\le T}$ *such that the component* $(Y_s^{t,x})_{t\le s\le T}$ *is bounded.*
(ii) If moreover, the function $(t,\, x) \in [0,\, T] \times I\!R^{d+1} \longmapsto v(t,\, x) := Y_t^{t,x}$ *is continuous, then it is a* L^p*-viscosity solution of the PDE (3.2).*

Proof. (i) Thanks to Remark 3.5 of [20], it is enough to prove existence and uniqueness for the BSDE,

$$Y_s^{t,x} = H(X_T^{t,x}) + \int_s^T g(u,\, X_u^{t,x},\, Y_u^{t,x})du - \int_s^T Z_u^{t,x} dB_u, t \le s \le T.$$

But, this can be proved by usual arguments of BSDEs. For instance, it's obvious that uniqueness holds under $(C1)$, and, we can prove the existence of the solution by using a Picard type approximation.
(ii) We only prove that v is L^p- viscosity sub-solution. The proof for super-solution can be performed similarly. For $\varphi \in W_{p,loc}^{1,2}\big([0,\, T] \times \mathrm{I\!R}^{d+1},\, \mathrm{I\!R}\big)$, let $(\widehat{t},\, \widehat{x}) \in [0,\, T] \times \mathrm{I\!R}^{d+1}$ be a point which is a local maximum of $v - \varphi$. Since $p > d+2$, then φ has a continuous version which we consider bellow. We assume without loss of generality that $v(\widehat{t},\, \widehat{x}) = \varphi(\widehat{t},\, \widehat{x})$. Following Cafarelli et al., assume that there exists $\varepsilon_0,\, r_0 > 0$ such that

$$\begin{cases} \frac{\partial \varphi}{\partial s}(s,\, x_1,\, x_2) + \overline{L}(x_1,\, x_2)\varphi(s, x_1,\, x_2) + g(s,\, x_1,\, x_2,\, v(s,\, x_1,\, x_2)) < -\varepsilon_0, \lambda\text{-}a.s \text{ in } B_{r_0}(\widehat{t},\, \widehat{x}) \\ \\ v(s,\, x) \le \varphi(s,\, x) + \varepsilon_0(s - \widehat{t}\,),\ \lambda\text{-}a.s \text{ in } B_{r_0}(\widehat{t},\, \widehat{x}) \end{cases}$$

Let $A_0 \in B_{r_0}(\widehat{t},\, \widehat{x})$ be a set of positive measure such that $(\widehat{t},\, \widehat{x}) \in A_0$. Define

$$\tau = \inf\big\{ s \ge \widehat{t}; \quad (s,\, X_s^{t,x}) \notin A_0 \big\} \wedge (\widehat{t} + r_0)$$

The process $(\overline{Y}_s,\, \overline{Z}_s) = ((Y_{s\wedge\tau}^{t,x}),\, \mathbf{1}_{[0,\,\tau]}(s)(Z_s^{t,x}))_{s\in[\widehat{t},\,\widehat{t}+r_0]}$ solves then the BSDE

$$\overline{Y}_s = v_l(\tau,\, X_\tau^{t,x}) + \int_s^{\widehat{t}+r_0} \mathbf{1}_{[0,\,\tau]}(u) g(u,\, X_u^{t,x},\, v(u,\, X_u^{t,x}))du - \int_s^{\widehat{t}+r_0} \overline{Z}_u dM_u^{X^{t,x}},\ s \in [\widehat{t},\, \widehat{t} + r_0].$$

On other hand, setting $\psi(s,\, x) = \varphi(s,\, x) + \varepsilon_0(s - \widehat{t}\,)$, we have by Itô-Krylov's formula that the process $(\widehat{Y}_s,\, \widehat{Z}_s) = \big(\psi(s,\, X_{s\wedge\tau}^{t,x}),\ \mathbf{1}_{[0,\,\tau]}(s)\nabla\varphi(s,\, X_s^{t,x})\big)_{s\in[\widehat{t},\,\widehat{t}+r_0]}$ solves the BSDE

$$\widehat{Y}_s = \psi(\tau,\, X_\tau^{t,x}) - \int_s^{\widehat{t}+r_0} \mathbf{1}_{[0,\,\tau]}(u)[\varepsilon_0 + (\frac{\partial\varphi}{\partial u} + \overline{L}\varphi)(u,\, X_u^{t,x})]du - \int_s^{\widehat{t}+r_0} \widehat{Z}_u dM_u^{X^{t,x}}.$$

¿From the choice of τ, $(\tau,\, X_\tau^{t,x}) \in A_0$. Therefore $v(\tau,\, X_\tau^{t,x}) \le \psi(\tau,\, X_\tau^{t,x})$ and thanks to the comparison theorem [20], we deduce that $\overline{Y}_{\widehat{t}} < \widehat{Y}_{\widehat{t}}$, i.e $v_l(\widehat{t},\, \widehat{x}) < \varphi(\widehat{t},\, \widehat{x})$, which contradicts our hypothesis. ∎

Remark 5.2. (i) Whenever g does not depends on t; $v(t,\, x) = \tilde{Y}_0^x$ is a L^p-viscosity solution of the PDE

$$\begin{cases} \frac{\partial v}{\partial s}(s,\, x_1,\, x_2) = \overline{L}(x_1,\, x_2)v(s, x_1,\, x_2) + g(x_1,\, x_2,\, v(s,\, x_1,\, x_2)) \\ \\ v(0, x_1,\, x_2) = H(x_1,\, x_2),\ s > 0,\, x = (x_1,\, x_2) \in \mathrm{I\!R}^{d+1} \end{cases}$$

where $(X^x,\, \tilde{Y}^x_s,\, \tilde{Z}^x_s; \quad 0 \le s \le t)$, solves the following decoupled FBSDE

$$\begin{cases} X^x_s = x + \int_0^s \overline{b}(X^x_u)du + \int_0^s \overline{\sigma}(X^x_u)dB_u, \quad 0 \le s \le t. \\ \\ \tilde{Y}^x_s = H(X^x_t) + \int_s^t g(X^x_u,\, \tilde{Y}^x_u)du - \int_s^t \tilde{Z}^x_u dM^{X^x}_u, \quad 0 \le s \le t \end{cases}$$

(ii) Since f satisfies **(C)** and ρ is bounded, one can easily verify that $\overline{f}$ satisfies $(C1)$. Therefore, for a fixed positive t, the BSDE with data $(H(X^x_t),\, \overline{f})$ admit a unique solution $(Y^x_s,\, Z^x_s)_{0\le s\le t}$. Moreover, if the function $(t,\, x) \in \mathbb{R}_+ \times \mathbb{R}^{d+1} \mapsto v(t,\, x) = Y^x_0$ is continuous and hence, it is a L^p-viscosity solution of the PDE (1.5).

Under assumptions **(A), (B)**, the SDE (3.1) has a unique weak solution [16]. We then have,

Proposition 5.3. *(Continuity in law of the flow $x \mapsto X^x_\cdot$)*

Assume **(A), (B)**. *Let X^x_s be the unique weak solution of the SDE* (3.1)*, and*
$X^n_s := x_n + \int_0^s \overline{b}(X^n_u)du + \int_0^s \overline{\sigma}(X^n_u)dB_u,\ 0 \le s \le t$
Assume that x_n converges towards $x = (x^1,\, x^2) \in I\!R^{1+d}$. Then, $X^n \overset{law}{\Longrightarrow} X^x$.

Proof. Since $\overline{b}$ and $\overline{\sigma}$ satisfy **(A), (B)**, one can easily check that the sequence X^n is tight in $\mathcal{C}([0,\, t] \times \mathbb{R}^{d+1})$. By Prokhorov's theorem, there exists a subsequence (denoted also by X^n) which converges weakly to a process $\widehat{X}$. We shall show that $\widehat{X}$ is a weak solution of SDE (3.1).

• *Step 1:* For every $\varphi \in C^\infty_c(\mathbb{R}^{1+d})$,

$$\forall u \in [0,\, t], \quad \varphi(\widehat{X}_u) - \int_0^u \overline{L}\varphi(\widehat{X}_v)dv \quad \text{ is a } \mathcal{F}^{\widehat{X}}\text{-martingale.}$$

We have need to show that for every $\varphi \in C^\infty_c(\mathbb{R}^{1+d})$, every $0 \le s \le u$ and every function Φ_s of $(X^{x_n}_r)_{0\le r<s}$ which is bounded and continuous in the topology of the uniform convergence,

$$0 = \mathbb{E}\left\{[\varphi(X^{x_n}_u) - \varphi(X^{x_n}_s) - \int_s^u \overline{L}\varphi(X^{x_n}_v)dv]\Phi_s(X^{x_n}_\cdot)\right\}$$

$$\overset{n}{\longrightarrow} \mathbb{E}\left\{[\varphi(\widehat{X}_u) - \varphi(\widehat{X}_s) - \int_s^u \overline{L}\varphi(\widehat{X}_v)dv]\Phi_s(\widehat{X}_\cdot)\right\}$$

Indeed, since φ, Φ are continuous functions and $\overline{L}$ is continuous out of the set $\{x_1 = 0\}$, similar argument as that developed in the proof of Lemma 4.7 gives

$$[\varphi(X^{x_n}_u) - \varphi(X^{x_n}_s) - \int_s^u \overline{L}\varphi(X^{x_n}_v)dv]\Phi_s(X^{x_n}_\cdot) \overset{law}{\longrightarrow} [\varphi(\widehat{X}_u) - \varphi(\widehat{X}_s) - \int_s^u \overline{L}\varphi(\widehat{X}_v)dv]\Phi_s(\widehat{X}_\cdot)$$

Since φ, Φ are bounded functions and $\sup_n \mathbb{E}(\sup_{s\in[0,\,t]} |X^{x_n}|^2) < \infty$, the result follows by the uniform integrability criterium. Hence, $\mathbb{E}\left\{[\varphi(\widehat{X}_u) - \varphi(\widehat{X}_s) - \int_s^u \overline{L}\varphi(\widehat{X}_v)dv]\Phi_s(\widehat{X}_\cdot)\right\} = 0$ and therefore $\varphi(\widehat{X}_u) - \varphi(\widehat{X}_s) - \int_s^u \overline{L}\varphi(\widehat{X}_v)dv$ is a $\mathcal{F}^{\widehat{X}}$-martingale.

•*Step 2:* From *step 1*, there exists a $\mathcal{F}^{\widehat{X}}$-Brownian motion $\widehat{B}$ such that,

$$\widehat{X}_s = x + \int_0^s \overline{b}(\widehat{X}_u)du + \int_0^s \overline{\sigma}(\widehat{X}_u)d\widehat{B}_u, \quad 0 \le s \le t.$$

Weak uniqueness for SDE (3.1) allows us to deduce that $X^{x_n} \overset{law}{\Longrightarrow} X^x$. ■

Proposition 5.4. *Assume* **(A), (B), (C)**. *Then,*

(*i*) $\lim_{\varepsilon\to 0} Y_0^\varepsilon = Y_0^{(t,x)}$.

(*ii*) *The map* $(t,x) \longmapsto Y_0^{t,x}$ *is continuous.*

(*iii*) *For* $p > d+2$, *the function* $v(t,x) := Y_0^{t,x}$ *is a* L^p*-viscosity solution to the PDE (1.5).*

Proof. (*i*) Let Y be the limit process defined in Proposition (4.3). Since Y_0^ε and Y_0 are deterministic, it is enough to prove that $\lim_{\varepsilon\to 0}\mathbb{E}(Y_0^\varepsilon) = \mathbb{E}(Y_0)$. We have,

$$\begin{cases} Y_0^\varepsilon = H(X_t^\varepsilon) + \int_0^t f(\overline{X}_u^\varepsilon,\, X_u^{2,\varepsilon},\, Y_u^\varepsilon)du - M_t^\varepsilon \\ \\ Y_0 = H(X_t) + \int_0^t \overline{f}(X_u, Y_u)du - M_t \end{cases}$$

¿From Jakubowski [12], the projection: $y \mapsto y_t$ is continuous in the **S**-topology. We then deduce that Y_0^ε converges towards Y_0 in distribution. Since Y_0^ε and Y_0 are bounded, then $\lim_{\varepsilon\to 0}\mathbb{E}(Y_0^\varepsilon) = \mathbb{E}(\overline{Y}_0)$.

(*ii*) Let $(t_n,\, x_n) \to (t,\, x)$. We assume that $t > t_n > 0$. We have,

$$Y_s^{t_n,x_n} \;=\; H(X_{t_n}^{x_n}) + \int_s^{t_n} \overline{f}(X_u^{x_n},\, Y_u^{t_n,x_n})du - \int_s^{t_n} Z_u^{t_n,x_n} dM_u^{X^{x_n}},\ 0 \le s \le t_n, \quad (5.1)$$

where $X^{x_n} \overset{law}{\Rightarrow} X^x$.
Since H is a bounded continuous function and $\overline{f}$ satisfies $(C1)$, one can easily show that the sequence $\{(Y^{t_n,x_n},\, \int_0^{\cdot} 1_{[s,t_n]}(u) Z_u^{x_n} dM_u^{X^{x_n}})\}_{n\in\mathbb{N}^*}$ is tight in $\mathcal{D}([0,\, t]\times\mathbb{R}\times\mathbb{R})$.
Let us rewrite the equation (5.1) as follows,

$$\begin{aligned} Y_s^{t_n,x_n} \;&=\; H(X_{t_n}^{x_n}) + \int_s^{t} \overline{f}(X_u^{x_n},\, Y_u^{t_n,x_n})du - \int_s^{t} 1_{[s,t_n]}(u) Z_u^{t_n,x_n} dM_u^{X^{x_n}} \\ &-\; \int_{t_n}^{t} \overline{f}(X_u^{x_n},\, Y_u^{t_n,x_n})du,\ 0 \le s \le t. \\ &=\; A_n^1 + A_n^2 \end{aligned} \quad (5.2)$$

• *Convergence of* A_n^2
One has $\mathbb{E}\left|\int_{t_n}^t \overline{f}(X_u^{x_n},\, Y_u^{t_n,x_n})du\right| \le K(|x|)|t-t_n|$. Hence A_n^2 tends to zero in probability.
• *Convergence of* A_n^1
Denote by $(Y',\, M')$ the weak limit of $\{(Y^{t_n,x_n},\, \int_0^{\cdot} 1_{[s,t_n]}(u) Z_u^{x_n} dM_u^{X^{x_n}})\}_{n\in\mathbb{N}^*}$. In view of Lemma 4.7, one has $\displaystyle\int_s^t \overline{f}(X_u^{x_n},\, Y_u^{t_n,x_n})du \overset{law}{\Longrightarrow} \int_s^t \overline{f}(X_u^x,\, Y'_u)du$.

Passing to the limit in (5.2), we obtain that

$$Y'_s \;=\; H(X_t^x) + \int_s^t \overline{f}(X_u^x,\, Y'_u)du - (M'_t - M'_s),\ s \in [0,\, t] \cap D^c.$$

The uniqueness of the considered BSDE ensures that $\forall s \in [0,\, t]$, $Y'_s = Y_s^{t,x}$ $\mathbb{P}$-ps. Hence $Y^{t_n,x_n} \overset{law}{\Rightarrow} Y^{t,x}$. As in (*i*), one derive that $Y_0^{t_n,x_n} \overset{law}{\Rightarrow} Y_0^{t,x}$ which yields to the continuity of $Y_0^{t,x}$.
Assertion (*iii*) follows from Remark 5.2. ∎

Remark As in KK, we can take W instead of $\widetilde{W}$.

A Appendix: S-topology

The **S**-topology has been introduced by Jakubowski ([12], 1997) as a topology defined on the Skorohod space of càdlàg functions: $\mathcal{D}([0,\,T];\,\mathbb{R})$. This topology is weaker than the Skorohod topology but tightness criteria are easier to establish. These criteria are the same as the one used in Meyer-Zheng topology, ([18], 1984).
Let $N^{a,\,b}(z)$ denotes the number of up-crossing of the function $z \in \mathcal{D}([0,\,T];\,\mathbb{R})$ in a given level $a < b$. We recall some facts about the **S**-topology.

Proposition A.1. *(A criteria for S-tight). A sequence $(Y^\varepsilon)_{\varepsilon>0}$ is S-tight if and only if it is relatively compact on the S-topology.*
Let $(Y^\varepsilon)_{\varepsilon>0}$ be a family of stochastic processes in $\mathcal{D}([0,\,T];\,\mathbb{R})$. Then this family is tight for the S-topology if and only if $(\|Y^\varepsilon\|_\infty)_{\varepsilon>0}$ and $(N^{a,\,b}(Y^\varepsilon))_{\varepsilon>0}$ are tight for each $a<b$.

Let $(\Omega,\,\mathcal{F},\,\mathbb{P},\,(\mathcal{F}_t)_{t\geq 0})$ be a stochastic basis. If $(Y)_{0\leq t\leq T}$ is a process in $\mathcal{D}([0,\,T];\,\mathbb{R})$ such that Y_t is integrable for any t, *the conditional variation of* Y is defined by

$$CV(Y) = \sup_{0\leq t_1<\ldots<t_n=T,\, partition\, of\, [0,\,T]} \sum_{i=1}^{n-1} \mathbb{E}[|\mathbb{E}[Y_{t_{i+1}} - Y_{t_i} \mid \mathcal{F}_{t_i}]|].$$

The process is call *quasimartingale* if $CV(Y) < +\infty$. When Y is a $\mathcal{F}_t$-martingale, $CV(Y) = 0$. A variation of Doob inequality (cf. lemma 3, p.359 in Meyer and Zheng, 1984, where it is assumed that $Y_T = 0$) implies that

$$\mathbb{P}\left[\sup_{t\in[0,\,T]} |Y_t| \geq k\right] \leq \frac{2}{k}\left(CV(Y) + \mathbb{E}\left[\sup_{t\in[0,\,T]} |Y_t|\right]\right),$$

$$\mathbb{E}\left[N^{a,\,b}(Y)\right] \leq \frac{1}{b-a}\left(|a| + CV(Y) + \mathbb{E}\left[\sup_{t\in[0,\,T]} |Y_t|\right]\right).$$

It follows that a sequence $(Y^\varepsilon)_{\varepsilon>0}$ is S-tight if

$$\sup_{\varepsilon>0}\left(CV(Y^\varepsilon) + \mathbb{E}\left[\sup_{t\in[0,\,T]} |Y_t^\varepsilon|\right]\right) < +\infty.$$

Theorem A.2. *Let $(Y^\varepsilon)_{\varepsilon>0}$ be a S-tight family of stochastic process in $\mathcal{D}([0,\,T];\,\mathbb{R})$. Then there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ decreasing to zero, some process $Y \in \mathcal{D}([0,\,T];\,\mathbb{R})$ and a countable subset $D \in [0,\,T]$ such that for any n and any $(t_1,\,...,\,t_n) \in [0,\,T]\backslash D$,*

$$(Y_{t_1}^{\varepsilon_k},\,...,\,Y_{t_n}^{\varepsilon_k}) \xrightarrow{\mathcal{D}ist} (Y_{t_1},\,...,\,Y_{t_n})$$

Remark A.3. The projection :$\pi_T\ y \in (\mathcal{D}([0,\,T];\,\mathbb{R}),\,S) \mapsto y(T)$is continuous (see Remark 2.4, p.8 in Jakubowski,1997), but $y \mapsto y(t)$ is not continuous for each $0 \leq t \leq T$.

Lemma A.4. *Let $(U^\varepsilon,\,M^\varepsilon)$ be a multidimensional process in $\mathcal{D}([0,\,T];\,\mathbb{R}^p)$ $(p \in \mathbb{N}^*)$ converging to $(U,\,M)$ in the S-topology. Let $(\mathcal{F}_t^{U^\varepsilon})_{t\geq 0}$ (resp. $(\mathcal{F}_t^U)_{t\geq 0}$) be the minimal complete admissible filtration generated by U^ε (resp. U). We assume moreover that. for every $T>0$, $\sup_{\varepsilon>0} \mathbb{E}\left[\sup_{0\leq t\leq T} |M_t^\varepsilon|^2\right] < C_T$.*
If M^ε is a $\mathcal{F}^{U^\varepsilon}$-martingale and M is $\mathcal{F}^U$-adapted, then M is a $\mathcal{F}^U$-martingale.

Lemma A.5. *Let $(Y^\varepsilon)_{\varepsilon>0}$ be a sequence of process converging weakly in $\mathcal{D}([0,T];\, I\!R^p)$ to Y. We assume that $\sup_{\varepsilon>0} I\!E\left[\sup_{0\le t\le T}|Y_t^\varepsilon|^2\right] < +\infty$. Hence, for any $t\ge 0$, $E\left[\sup_{0\le t\le T}|Y_t|^2\right] < +\infty$.*

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